International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Symmetric Chain Partitions of Orthocomplemented Posets

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Received October 18, 1994

It is shown that any orthocomplemented poset P of finite width admits a chain partition of cardinality $2\lfloor \frac{2}{3}\,\mathrm{width}(P) \rfloor$ which is symmetrical with respect to the orthocomplement. This cardinality is the best possible.

1. INTRODUCTION

Let $(P, <)$ be a poset. A set of pairwise comparable or pairwise incomparable elements of P is called a chain or an antichain, respectively. A chain partition of P is a set $\{C_1, \ldots, C_k\}$ of subsets of P, each of which is a chain, such that the union $\bigcup_{i=1}^{k} C_i = P$. Since each chain intersects each antichain in at most one element, there must be at least as many chains in any chain partition as there are elements in an antichain. A classic result of combinatorial order theory [See Dilworth (1950) and Mirsky (1971); for some applications also see Kaldewaij (1987), Larman *et al.* (1994), and Pach and Töröcsik (1994)] is as follows:

Theorem (Dilworth, 1950). The minimum cardinality of a chain partition of P equals the maximum cardinality of an antichain in P.

This common cardinality is called the width of P. In the following we always assume the width of the poset to be finite.

An orthocomplemented poset is a poset $(P, <)$ together with a self-mapping $\bar{\ }: P \to P$ which is an antisymmetry ($\bar{x} < \bar{y} \Leftrightarrow y < x$), an involution ($\bar{\bar{x}} = x$), and an orthocomplement ($\neg\exists x, y$: $x < y \wedge \bar{x} < y$). Normally (Flachsmeyer, 1988; Giuntini, 1991) also a global minimum 0 and a global maximum 1 are postulated (with the appropriate change of the orthocomple-

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0020-7748/95/0800-1241$07.50/0

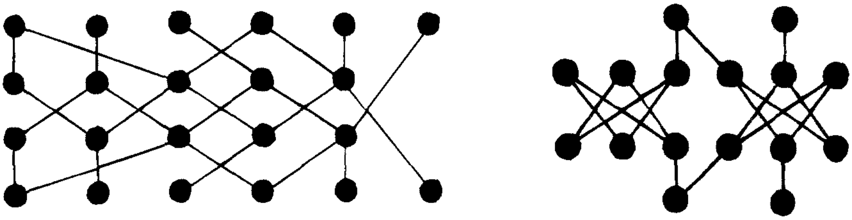

Fig. 1. Hasse diagrams of orthocomplemented posets.

mentarity axiom); we exclude them for technical reasons without loss of generality. In the following all Hasse diagrams of examples are drawn in such a way that the orthocomplement is given by a reflection along a horizontal line (Fig. 1).

Although the orthocomplement is a symmetry of the poset, it is generally not possible to find a minimum cardinality chain decomposition which is symmetric with respect to the orthocomplement, for the orthocomplement maps chains on disjoint chains, so any symmetric chain decomposition must have even cardinality. But there are orthocomplemented posets of odd width (Fig. 2, with chains of a Dilworth-decomposition marked).

Orthocomplemented posets constructed as in Fig. 3 show that $2\lfloor\frac{2}{3}\,\mathrm{width}(P)\rfloor$ chains may be necessary for a symmetric chain partition of P if $\mathrm{width}(P) \equiv 0 \bmod 3$. For $\mathrm{width}(P) \equiv -1$ or $1 \bmod 3$, one of the $\lfloor\frac{1}{3}\,\mathrm{width}(P)\rfloor$ components of this poset has to be replaced by one or two copies of the poset consisting of two complementary (incomparable) elements to reach this bound. It is the aim of this paper to show that this number is also sufficient.

Theorem. Each orthocomplemented poset admits a symmetric chain partition with at most $2\lfloor\frac{2}{3}\,\mathrm{width}(P)\rfloor$ chains.

Adding 0 and 1 turns these examples into orthomodular lattices.

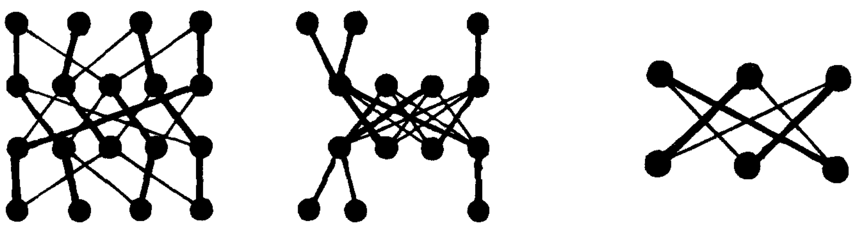

Fig. 2. Orthocomplemented posets of odd width.

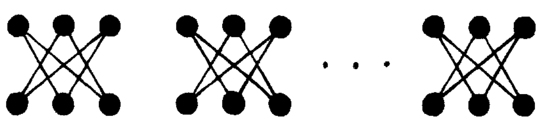

Fig. 3. The extremal orthocomplemented poset.

2. THE STRUCTURE OF MAXIMUM CARDINALITY ANTICHAINS

Each inclusion-maximal antichain X decomposes the poset P in a lower part $L(X) := \{p \in P | p \leq x \text{ for some } x \in X\}$ and an upper part $U(X) := \{p \in P | x \leq p \text{ for some } x \in X\}$, with $L(X) \cup U(X) = P$, $L(X) \cap U(X) = X$. This defines a partial order $\leq^*$ on the inclusion-maximal antichains by $X_1 \leq^* X_2$ iff $X_1 \subseteq L(X_2)$. If we restrict this to the set $\mathcal{M}(P)$ of maximum-cardinality antichains of P, it becomes a lattice. For, if $X_1, X_2 \in \mathcal{M}$ are maximum-cardinality antichains, then $X_1 \cup X_2$ is a height-2 poset, so $X_1 \cup X_2 = \inf(X_1, X_2) \cup \sup(X_1, X_2)$, where $\inf(X_1, X_2)$ and $\sup(X_1, X_2)$ denote the sets of minimal and maximal elements of $X_1 \cup X_2$, respectively. These sets are antichains, so they are at most of cardinality $|X_1| = |X_2|$, and since $X_1 \cap X_2 = \inf(X_1, X_2) \cap \sup(X_1, X_2)$, they must both be maximum-cardinality antichains. The orthocomplement induces an involutive antisymmetry of the lattice, which is not an orthocomplement of the lattice.

We are especially interested in those maximum-cardinality antichains $A \in \mathcal{M}$ with $A \leq^* \overline{A}$; we call them lower antichains. For each antichain $X \in \mathcal{M}$ the infimum $\operatorname{syminf}(X) := \inf(X, \overline{X})$ is a lower antichain, and for each $x \in X$ at least one of $x, \bar{x}$ is contained in $\operatorname{syminf}(X)$. Each lower antichain A decomposes the poset into the outer parts $L(A) \cup \overline{L(A)} = L(A) \cup U(\overline{A})$ and the middle part $\overline{L(\overline{A})} \cap L(\overline{A}) = U(A) \cap L(\overline{A})$. If $A \in \mathcal{M}$ is a lower antichain and $X \in \mathcal{M}$ is an antichain that is not contained in $L(A) \cup \overline{L(A)}$ (so there are $x \in X$, $a_1, a_2 \in A$ with $a_1 < x < \overline{a_2}$), then there is a lower antichain $\operatorname{cov}(A, X) := \operatorname{syminf}(\inf(\overline{A}, \sup(X, A)))$ such that $A \leq^* \operatorname{cov}(A, X)$ and $X \subseteq L(\operatorname{cov}(A, X)) \cup \overline{L(\operatorname{cov}(A, X))}$.

3. PARTIAL CHAIN DECOMPOSITIONS

Let P be a poset of width w. Let $\mathcal{C}$ denote the family of those w-tuples of nonempty chains $(C_1, \ldots, C_w)$ for which $\bigcup_{i=1}^{w} C_i$ is a lower order ideal of P and that have the property that for each transversal $\{t_1, \ldots, t_w\} \subseteq P$, $t_i \in C_i$, there is a lower antichain $\{a_1, \ldots, a_i\} \in \mathcal{M}$, $a_i \in C_i$, such that $t_i \leq a_i$ (transversal covering property). This family is not empty; if $A \in \mathcal{M}$ is a lower antichain, then any Dilworth-decomposition of $L(A)$ belongs to $\mathcal{C}$.

This family is ordered by componentwise inclusion, and each chain in it possesses an upper bound. It follows by Zorn's Lemma that $\mathcal{C}$ contains a maximal element $(C_1^*, \ldots, C_w^*)$, which we keep fixed. This decomposes some lower order ideal $L^* := \bigcup_{i=1}^{w} C_i^*$. For each lower antichain $A \in \mathcal{M}$, $A \subseteq L^*$ this induces a special Dilworth-decomposition $C_1(A) \cup \cdots \cup C_w(A)$ of $L(A)$ for which $C_i(A) \subseteq C_i^*$. By the transversal covering property we may select a $\leq^*$-chain $\mathcal{A}$ of lower antichains contained in L^* such that $C_i^* = \bigcup_{A \in \mathcal{A}} C_i(A)$.

Suppose now that there is some maximum-cardinality antichain $X \in \mathcal{M}$ such that $X \not\subseteq L^* \cup \overline{L^*}$. For each $A \in \mathcal{A}$ the set $\text{cov}(A, X) \cap (X \cup \overline{X})$ is one of the finitely many nonempty subsets of $(X \cup \overline{X})$, so there is a subchain $\mathcal{A}_{\text{sub}} \subset \mathcal{A}$ with $C_i^* = \bigcup_{A \in \mathcal{A}_{\text{sub}}} C_i(A)$ and $\text{cov}(A, X) \cap (X \cup \overline{X})$ is constant for $A \in \mathcal{A}_{\text{sub}}$. Then $\mathcal{A}_X := \{\text{cov}(A, X) | A \in \mathcal{A}_{\text{sub}}\}$ is again a $\leq^*$-chain, and each element of $\mathcal{A}_X$ defines a Dilworth-decomposition $C_1(\text{cov}(A, X)) \cup \cdots \cup C_w(\text{cov}(A, X))$ which may be chosen such that $C_i(A) \subseteq C_i(\text{cov}(A, X)) \cap L^* \subset C_i^*$. This implies the existence of an element $(C_1^{**}, \ldots, C_w^{**}) \in \mathcal{C}$ with $C_i^* \subseteq C_i^{**}$ and $\bigcup_{i=1}^{w} C_i^{**} \supseteq L^* \cup \bigcup_{A \in \mathcal{A}_{\text{sub}}} \text{cov}(A, X) \supset L^*$, a contradiction to the maximality of $(C_1^*, \ldots, C_w^*)$. So each maximum-cardinality antichain is contained in $L^* \cup \overline{L^*}$.

4. PROOF OF THE THEOREM

The theorem is proved by induction on the width. It is certainly true for width 0, 1, 2, since only the empty poset is an orthocomplemented poset of width at most 1, which admits a symmetrical partition in 0 chains, and any orthocomplemented poset of width 2 consists of two complementary chains, so it is symmetrically partitioned in 2 chains. (At this point including 0 and 1 would cause exceptional configurations.)

Let now a poset P of width $w \geq 3$ be given. We take a maximum partial chain decomposition $(C_1^*, \ldots, C_w^*)$ as described before, together with the generating $\leq^*$-chain of lower antichains $\mathcal{A}$.

We define for each of the lower antichains $A \in \mathcal{A}$ a bipartite graph with vertexset $\{1, \ldots, w\} \cup \{\bar{1}, \ldots, \bar{w}\}$ and edgeset $E(A) := \{(i, \bar{j}) | \max(C_i(A)) \leq \min(\overline{C_j(A)})\}$. Since the lower antichains are of maximum cardinality, we have for each subset $S \subseteq \{1, \ldots, w\}$

$$\left|\{\bar{t} | \{s, \bar{t}\} \in E(A), s \in S\}\right| \geq |S|$$

i.e., the graph satisfies Hall's condition (Mirsky, 1971). For $A_1 \subset A_2$ we have $E(A_2) \subset E(A_1)$, so $\{E(A) | A \in \mathcal{A}\}$ is a decreasing sequence of finite sets; therefore the limit $\bigcap_{A \in \mathcal{A}} E(A)$ still satisfies Hall's condition. By the matching theorem we find a permutation μ of $\{1, \ldots, w\}$ such that $\{i, \overline{\mu(i)}\} \in E(A)$ for all i, A. So $C_i^* \cup \overline{C_{\mu(i)}^*}$ is a chain for each i, which gives a Dilworth-decomposition of $L^* \cup \overline{L^*}$.

Consider now the poset $P' := P \backslash (C_1^* \cup \overline{C_{\mu(1)}^*})$. This is a poset of width $w - 1$, since any antichain of cardinality w in P is already contained in $L^* \cup \overline{L^*}$, so it must intersect each chain of this Dilworth-decomposition in one element. We can now find a Dilworth-decomposition of $P' = D_1 \cup \cdots \cup D_{w-1}$ which we may select in such a way that either $\overline{C_1} \subseteq D_1$ and $C_{\mu(1)} \subseteq D_2$ or even $\overline{C_1} \cup C_{\mu(1)} \subseteq D_1$. Then $P'' := P \backslash (D_1 \cup \overline{D_1} \cup D_2 \cup \overline{D_2}) \subseteq P' \backslash (D_1 \cup D_2)$ is an orthocomplemented poset of width at most $w - 3$. So we have found a partition $P = P'' \cup D_1 \cup \overline{D_1} \cup D_2 \cup \overline{D_2}$ in an orthocomplemented poset of width $w - 3$ and a symmetrical set of four chains, which completes the inductive proof.

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